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The Reduction of a Quantum System of Three Identical Particles on a Plane

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1 Introduction

In this report, a quantum center–of–mass system of three identical particles on a plane is considered, and reduced to a system which has less degrees of freedom. The reduction will be performed through two symmetry structures in the system, which are;

1. rotation of all particles about the origin makes no difference in the physical state of the system,

2. the system is indistinguishable when particles are exchanged, because all particles are identical.

Figure 1 shows the overall idea of reduction. As a practical application of the

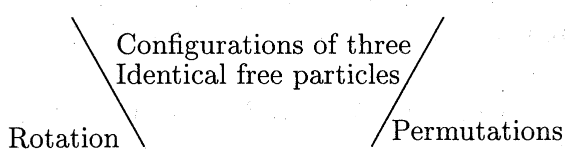

Figure 1: The configuration space admits the action of the rotation group and of the symmetric group, which are to the left and to the right respectively.

theory, a free three–particle quantum planar system is considered. On the basis of the rotational symmetry, the center–of–mass system for the planar three-body system is made into a principal fiber bundle $\dot{\mathrm{R}}^4 \longrightarrow \dot{\mathrm{R}}^3$ with structure group $\mathrm{SO}(2)$, where the action of the group is to the left, and the

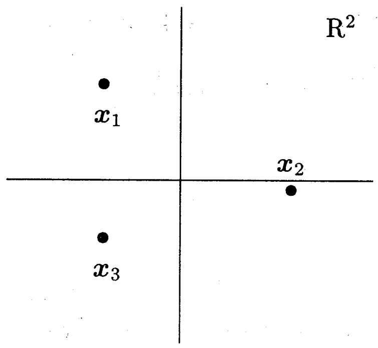

Figure 2: Three particles are to move around on the plane. All particles are labeled, so that exchanges of particles are trackable.

dot symbol indicates that the origin is removed from the space in question. A similar process should be taken for the particle exchange symmetry, which may be carried out in terms of the symmetric group S_3. A point to make here is that the theory should apply to a system containing any number of particles, and of course can do in three dimensions, too. The reason why this particular example is chosen is because for systems with four or more particles, the symmetric group S_n arriving from particle exchanges gets rapidly more difficult to treat in an explicit manner, and we feel that $n = 3$ for the number of particles is comfortable to present the idea.

2 Configuration Space and Jacobi Vectors

Suppose there are three particles on a plane, each with position vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$, and masses m_1, m_2, and m_3, respectively. The particles are constrained to move on the plane, so $\boldsymbol{x}_j \in \mathrm{R}^2$. The set of all possible particle positions is identified with $X \cong \mathrm{R}^{2\times 3}$, which consists of ordered triples of position vectors $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. The position vector of particle 1 is entered into the left most slot in the brackets. Figure 2 illustrates the spreaded particles on the plane. Each particle is labeled for the time being.

Since equations of motion are not yet given, the particles have no prescribed motions. The purpose of the current discussion is to give a rough idea of the space that particles can lie in.

Given the space X with labeled particles, one can consider two fundamental motions traced by the particles, one of which is the translation;

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \longmapsto (\boldsymbol{x}_1 + \boldsymbol{a}, \boldsymbol{x}_2 + \boldsymbol{a}, \boldsymbol{x}_3 + \boldsymbol{a}), \quad \boldsymbol{a} \in \mathrm{R}^2, \tag{1}$$

and the other is the rotation

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \longmapsto (g\boldsymbol{x}_1, g\boldsymbol{x}_2, g\boldsymbol{x}_3), \quad g \in SO(2). \tag{2}$$

The space X is endowed with the inner product $K : X \times X \longrightarrow \mathrm{R}$,

$$K(x, y) = \sum_{j=1}^{3} m_j(\boldsymbol{x}_j, \boldsymbol{y}_j), \quad x, y \in X, \tag{3}$$

where $(\boldsymbol{x}, \boldsymbol{y})$ denotes the standard inner product on R^2. Note that this metric incorporates mass, which will cause later the absence of m, a mass factor, in the Schrödinger equation. This is because the coordinate system that is to be produced on the basis of this metric contains mass already.

We shall now focus on the center–of–mass system, which means that the center of mass of the particles remains fixed at the origin;

$$\sum_{1}^{3} m_j \boldsymbol{x}_j = 0, \tag{4}$$

and this will imply that no action of translations is possible. We shall denote this center–of–mass system by

$$X_0 = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \in X | \sum_{j=1}^{3} m_j \boldsymbol{x}_j = 0\} \tag{5}$$

The space X has its natural orthonormal basis $(\boldsymbol{e}_1, 0, 0), (\boldsymbol{e}_2, 0, 0), \ldots,$ conforming to the inner product defined by (3), but this does not mean that the subspace $X_0 \subset X$ has the same basis. Here $\boldsymbol{e}_j$ are the standard basis vectors for R^2. Any basis for X_0 must satisfy the condition (4) and so must be manually constructed by the Gram–Schmidt process. We note that the pair $(-m_2\boldsymbol{e}_1, m_1\boldsymbol{e}_1, 0)$ and $(-m_2\boldsymbol{e}_2, m_1\boldsymbol{e}_2, 0)$ satisfies (4), and that these vectors are orthogonal with respect to the metric (3). Normalizing theses vectors and using them as the seeds for the process, we find that a suitable set of orthonormal basis vectors shall be given by

$$f_1 = N_1(-m_2\boldsymbol{e}_1, m_1\boldsymbol{e}_1, 0), \tag{6}$$
$$f_2 = N_1(-m_2\boldsymbol{e}_2, m_1\boldsymbol{e}_2, 0), \tag{7}$$
$$f_3 = N_2(-m_3\boldsymbol{e}_1, -m_3\boldsymbol{e}_1, (m_1+m_2)\boldsymbol{e}_1), \tag{8}$$
$$f_4 = N_2(-m_3\boldsymbol{e}_1, -m_3\boldsymbol{e}_2, (m_1+m_2)\boldsymbol{e}_2), \tag{9}$$

where N_j are the normalizing factors explicitly given by

$$N_1 = (m_1 m_2 (m_1+m_2))^{-1/2}, \tag{10}$$
$$N_2 = (m_3(m_1+m_2)(m_1+m_2+m_3))^{-1/2}. \tag{11}$$

Since $f_j, j = 1, \ldots, 4$ are basis vectors on X_0, any $x \in X_0$ can be represented as a linear combination of f_j's;

$$x = \sum_{j=1}^{4} q_j f_j, \qquad q_j = K(x, f_j), \tag{12}$$

where q_j are the coefficients, and define the new coordinate system for X_0, and in what follows we shall call the space X_0 the configuration space. The coordinate system (q_j) will reappear later, but for the time being an alternative system is considered, as the new system is more suitable for dealing with particle exchanges.

The space X_0 is isomorphic to R^4 and also to $\mathrm{R}^2 \times \mathrm{R}^2$, the set of pairs of vectors in R^2. We define the pair of two vectors as follows;

$$\boldsymbol{r}_1 = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 = \sqrt{\frac{m_1 m_2}{m_1+m_2}}(\boldsymbol{x}_2 - \boldsymbol{x}_3), \tag{13}$$
$$\boldsymbol{r}_2 = q_3\boldsymbol{e}_1 + q_4\boldsymbol{e}_2 = \sqrt{\frac{m_3(m_1+m_2)}{m_1+m_2+m_3}}\left(\boldsymbol{x}_3 - \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1+m_2}\right). \tag{14}$$

The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are called the Jacobi vectors. One vector is pointing along the line between particles 1 and 2, while the other is pointing along the line between particle 3 and the center of mass of particles 1 and 2. Figure 3 illustrates the visual view of the idea.

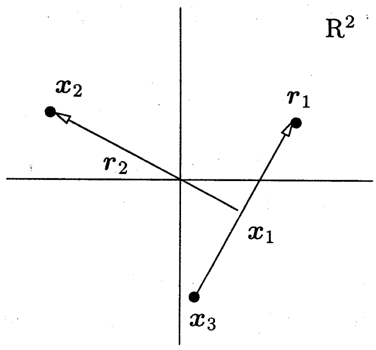

Figure 3: Illustrating the Jacobi vectors r_1 and r_2 as seen in Eqs. (13-14). r_1 points along the line joining particles 1 and 2, while r_2 points along the line joining particles 3 and the center of mass of particles 1 and 2. Note that the arrow lengths are *not* drawn to scale.

3 Exchanges of Particles

It it now important to recall that one of the symmetries that is utilized to perform the reduction is the indistinguishibility of configurations arising from exchanges of identical particles. Thus in this section, we make all particles identical and without loss of generality put $m_j = 1, j = 1, 2, 3$. Then the Jacobi vectors defined in Eqs. (13-14) become

$$\begin{aligned} \boldsymbol{r}_1 &= \frac{1}{\sqrt{2}}(\boldsymbol{x}_2 - \boldsymbol{x}_1), && (15) \\ \boldsymbol{r}_2 &= \sqrt{\frac{2}{3}}\left(\boldsymbol{x}_3 - \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2}\right). && (16) \end{aligned}$$

If, for example, particles 1 and 2 are exchanged, then the configuration undergoes a change

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \longmapsto (\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_3). \tag{17}$$

This can be more generalized. Since any combination of exchanges of particles can be expressed as a parmutation map $\sigma \in S_3$, where S_3 is the symmetric group of order six, the change the configuration takes is expressed as

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \longmapsto (\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \boldsymbol{x}_{\sigma(3)}). \tag{18}$$

The Jacobi vectors associated with $(\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \boldsymbol{x}_{\sigma(3)})$ are then expressed as

$$\boldsymbol{r}'_1 = \frac{1}{\sqrt{2}}(\boldsymbol{x}_{\sigma(2)} - \boldsymbol{x}_{\sigma(1)}), \tag{19}$$

$$\boldsymbol{r}'_2 = \sqrt{\frac{2}{3}}\left(\boldsymbol{x}_{\sigma(3)} - \frac{\boldsymbol{x}_{\sigma(1)} + \boldsymbol{x}_{\sigma(2)}}{2}\right). \tag{20}$$

If particles are exchanged in the manner of (17), then visually the direction of the arrow drawn to represent $\boldsymbol{r}_1$ is reversed. Bearing this in mind, one soon realizes that any combination of particle exhcnages can be represented by a linear transformation of Jacobi vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This will imply that this center-of-mass system of three identical particles admits the action of S_3 to the right. As in the case of example (17), one finds that the new pair of Jacobi vectors after the particle exhcnage is given by

$$(\boldsymbol{r}'_1, \boldsymbol{r}'_2) = (\boldsymbol{r}_1, \boldsymbol{r}_2)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

If one works through all possible combinations of exchanges, one will find the correspondence of permutations with linear transformation matrices. The graphical representation of this is simply given in Figure 4 indicating which transformation takes the reference Jacobi vectors to which pair of new Jacobi vectors. We have to note here that we are dealing with the right action of matrices, which is expressed as

$$(\boldsymbol{r}_1, \boldsymbol{r}_2)A = (\sum_{j=1}^{2} \boldsymbol{r}_j a_{j1}, \sum_{j=1}^{2} \boldsymbol{r}_j a_{j2}) \quad \text{for} \quad A = (a_{ji}), \tag{22}$$

and has the property

$$((\boldsymbol{r}_1, \boldsymbol{r}_2)A)B = (\boldsymbol{r}_1, \boldsymbol{r}_2)(BA). \tag{23}$$

Hence, the representation of S_3, $\rho:\ S_3 \to GL(2, \mathrm{R})$ that is required to satisfy $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for $g_1, g_2 \in S_3$, must act on X_0 in the manner

$$(\boldsymbol{r}_1, \boldsymbol{r}_2) \mapsto (\boldsymbol{r}_1, \boldsymbol{r}_2)\rho(g)^{-1}, \quad g \in S_3. \tag{24}$$

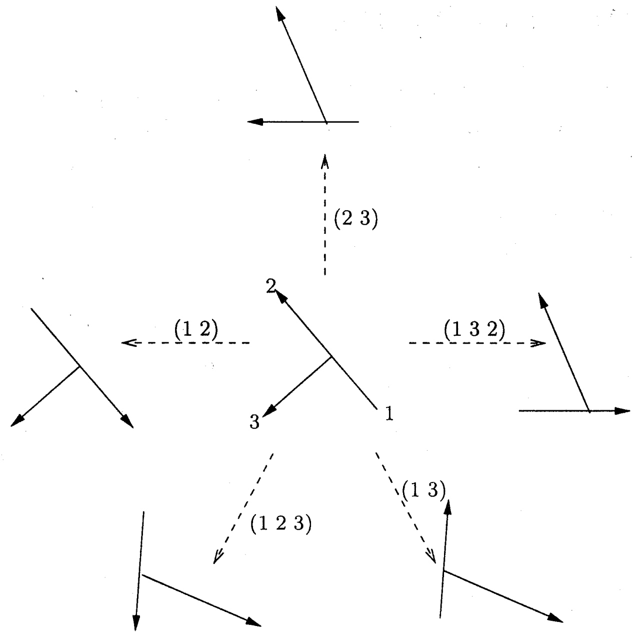

Figure 4: This diagram represents the graphical view of all possible particle exchanges. Numbers in brackets are the elements of permutations from S_3.

A straightforward calculation then provides

$$\begin{aligned}
\rho(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho(1\,2) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\rho(1\,2\,3) &= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, & \rho(1\,3\,2) &= \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \\
\rho(2\,3) &= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, & \rho(1\,3) &= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.
\end{aligned} \tag{25}$$

It is an easy matter to verify that the matrices in (25) form a discrete subgroup of O(2) which is isomorphic to the symmetric group S_3.

Implementing this for an n-particle system with $n \geq 4$ is not an easy task, as we have alluded in Introduction.

4 The Internal Space

What was established in Sec.3, was to describe the action of S_3 on the center-of-mass system, which comes from exchanges of particles. In this section, the symmetry due to the rotation is considered. Having removed the translational degrees of freedom, we can now consider the relative positions of the particles. If a given configuration of particles $x \in X_0$ is rotated about the origin and is found to fit another configuration $y \in X_0$, say, then x and y are said to be "similar" and there exists a linear transformation $g \in \mathrm{SO}(2)$ such that $x = gy$. For convenience, we forget the case where all particles collide at the origin, and consider the configuration space $\dot{X}_0 := X_0 - \{0\}$ in the following. Here the explicit expression of the g action is given by;

$$x = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \longmapsto (g\boldsymbol{x}_1, g\boldsymbol{x}_2, g\boldsymbol{x}_3) = gx, \quad g \in \mathrm{SO}(2), \;\; x \in \dot{X}_0, \tag{26}$$

and the "similarity" of x and y can be easily shown to be an equivalence relation;

$$x \sim y \quad \text{if and only if} \quad x = gy. \tag{27}$$

Then there exists the natural projection π from the configuration space to the quotient space;

$$\pi : \dot{X}_0 \longrightarrow M := \dot{X}_0/\mathrm{SO}(2), \tag{28}$$

which is defined to be

$$\pi(x) = [x], \quad x \in \dot{X}_0, \tag{29}$$

where $[x]$ denotes the equivalence class of x. The space M in (28) contains all possible inequivalent classes with respect to the equivalence relation (27). Physically, the elements of M express labels of different triangle shapes formed by $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ without counting the duplicated ones due to rotations about the origin.

In addition, M turns out to be a manifold which we shall call the internal or shape space, and $\dot{X}_0$ is made into a fiber bundle. A graphical view of this projection (28) is presented in figure 5 .

To elaborate the discussion, we wish to give the explicit form of the projection (29). Let $q = (q_1, q_2, q_3, q_4)$ denote points of the configuration

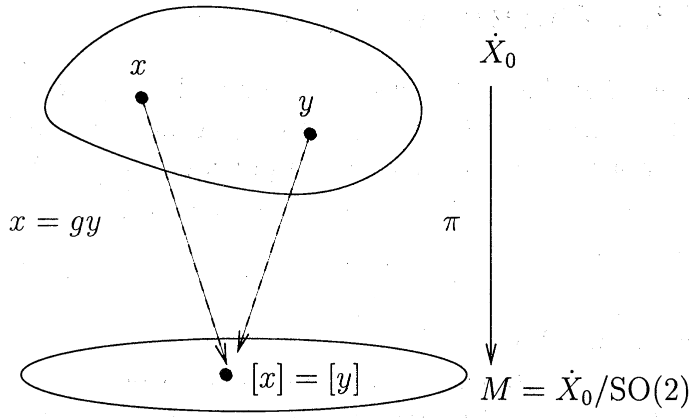

Figure 5: The projection π takes two equivalent configurations to the same equivalent class in M. In reality, x and y are on the same fiber.

space $\dot{X}_0$, just as was previously defined. We notice that X_0 can be identified with C^2 by introducing the complex vriables z_1, z_2 through

$$z_1 = q_1 + iq_2, \tag{30}$$
$$z_2 = q_3 + iq_4. \tag{31}$$

We work in terms of the new coordinates. On account of (13), (14), and (26) with $g = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, the action of SO(2) on C^2 turns out to be expressed as

$$z = (z_1, z_2) \longmapsto (e^{it}z_1, e^{it}z_2) = e^{it}z. \tag{32}$$

Here t is *not* considered as the temporal variable, but as a parameter of SO(2). With the identification $X_0 \cong \mathrm{C}^2$, the natural projection π is expressed as

$$\pi : (z_1, z_2) \longmapsto (\xi_1, \xi_2, \xi_3), \tag{33}$$

where

$$\xi_1 + i\xi_2 = 2z_1\bar{z}_2, \tag{34}$$
$$\xi_3 = |z_1|^2 - |z_2|^2. \tag{35}$$

It can be verified that the internal space M is diffeomorphic with $\dot{\mathrm{R}}^3$ which is R^3 with the origin removed;

$$M := \dot{X}_0/\mathrm{SO}(2) \cong \dot{\mathrm{R}}^3 := \mathrm{R}^3 - \{0\}. \tag{36}$$

5 The Action of S_3 on M

In Section 3, we have observed that the exchanges of identical particles give rise to the action of S_3 on X_0, and S_3 turned out to be represented as a discrete subgroup of O(2) consisting of six elements. With the identification $X_0 \cong \mathrm{C}^2$, the action of S_3 on X_0 is expressed as

$$(z_1, z_2) \mapsto (z_1, z_2)\rho(h)^{-1}, \quad h \in S_3, \tag{37}$$

as is seen from (24).

We now consider how this action behaves on the internal space M. For $h \in S_3$, the associated matrix $\tau(h)$ is defined by

$$[x\rho(h)^{-1}] := [x]\tau(h)^{-1}, \quad [x] \in M. \tag{38}$$

Note here that this definition is independent of the choice of representatives. From (34-35) together with (37- 38), the symmetric group S_3 is shown to act on M to the right;

$$(\xi_1, \xi_2, \xi_3) \mapsto (\xi_1, \xi_2, \xi_3)\tau(g)^{-1}, \quad g \in S_3, \tag{39}$$

where τ is a representation $\tau : S_3 \to \mathrm{GL}(3, \mathrm{R})$. A straightforward calculation shows that $\tau(S_3)$ forms a discrete subgroup of SO(3), which is expressed as

$$\begin{aligned}
\tau(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \tau(1\ 2) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\tau(1\ 2\ 3) &= \begin{pmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{pmatrix}, & \tau(1\ 3\ 2) &= \begin{pmatrix} -1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & -1/2 \end{pmatrix}, \\
\tau(2\ 3) &= \begin{pmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & -1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{pmatrix}, & \tau(1\ 3) &= \begin{pmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & -1 & 0 \\ -\sqrt{3}/2 & 0 & -1/2 \end{pmatrix}.
\end{aligned} \tag{40}$$

It is of interest to observe that the matrices acting on M have the unit determinant while those acting on X_0 have determinants either 1 or -1. At first sight, the dimension of matrices presented in (40) is 3×3, which is larger than those presented in (25), resulting an increase in the number of dimension by one. This seems not to fit the fact that the dimension of M

is less than that of $\dot{X}_0$. However, this is not a contradiction. While we have identified X_0 with $\mathrm{R}^{2\times 2}$, the set of Jacobi vectors, we are allowed to identify X_0 with R^4, the set of row vectors of length 4, so that we would have seen a discrete subgroup of GL(4, R) acting, and have been able to see an immediate reduction in the size of matrices. In fact, the O(2) action given in (24) proves to take the form

$$(q_1, q_2, q_3, q_4) \mapsto (q_1, q_2, q_3, q_4) \begin{pmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{pmatrix} \quad \text{for } \rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{41}$$

where $g \in S_3$ and I_2 denotes the 2×2 unit matrix. Thus we also see that the determinant of the 4×4 matrix in the above is the square of the determinat of the 2×2 matrix $\rho(g)$, that is, unity, so that the S_3 is represented as a discrete subgroup of SO(4).

6 Reduction by Rotation Symmetry

In this section, we present the reduction of the system of three free particles by rotation symmetry. The reduction goes through irrespective of whether all three particles are identical or not. The Schrödinger equation for free particles we consider here is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\nabla^2\psi \quad \text{with} \quad \nabla^2 = \sum_{j=1}^{4}\frac{\partial^2}{\partial q_j^2}. \tag{42}$$

In fact, since the oprator $\sum_{k=1}^{3}(1/m_k)(\partial/\partial \boldsymbol{x}_k)^2$ is the Laplacian with respect to the metric (3) on X, and since this metric is expressed as $\sum_{j=1}^{4} dq_j^2$ if restricted to the linear subspace X_0 of X, so that our Laplacian takes the form of $\nabla^2 = \sum_{j=1}^{4}\partial^2/\partial q_j^2$ in the coordinates (q_j). As is well known, this equation can be solved by Fourier transform with little difficulty, to give solution of the form

$$\psi(z,t) = \int_{\mathrm{C}^2} G(z,t;w,t_0)\psi_0(w,t_0)dw, \tag{43}$$

where G is the Green function, of which the explicit form is given by

$$G(z,t;w,t_0) = \left[\frac{1}{2\pi i\hbar(t-t_0)}\right]^2 \exp\left(\frac{i|z-w|^2}{2\hbar(t-t_0)}\right). \tag{44}$$

By $|z-w|^2$, we mean the Euclidian distance in C^2;

$$|z-w|^2 = |z_1-w_1|^2+|z_2-w_2|^2 \tag{45}$$

$$= \sum_{j=1}^{4}(q_j-p_j)^2, \tag{46}$$

where

$$w_1 = p_1+ip_2, \tag{47}$$

$$w_2 = p_3+ip_4, \tag{48}$$

and q_j's as previously defined in (30-31).

Since the Schrödinger equation (42) is invariant under the SO(2) action (32), the free particle system will be shown to be reduced to a system on the internal space M. Before proceeding with the reduction, we need a decomposition of $L^2(\mathrm{C}^2)$ with the SO(2) action. For $f \in L^2(\mathrm{C}^2)$ given, we consider a function $f(e^{is}z)$ with a parameter s, which can be expanded into the Fourier series

$$f(e^{is}z) = \sum_{m=-\infty}^{\infty} f_m(z)e^{ims}, \quad f_m(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{is}z)e^{-ims}ds. \tag{49}$$

In particular, we have, for $s=0$,

$$f(z) = \sum_{m=-\infty}^{\infty} f_m(z). \tag{50}$$

Note that the function f_m defined in (49) satisfies $f_m(e^{is}z) = e^{ims}f_m(z)$. Any function f on C^2 satisfying the equation

$$f(e^{is}z) = e^{ims}f(z) \tag{51}$$

is called ρ_m-equivariant, where ρ_m denotes a unitary irreducible representation of $\mathrm{SO}(2) \cong \mathrm{U}(1)$, $\rho_m(e^{is}) = e^{ims}$. On account of the invariance of the Lebesgue measure dz on C^2 under the SO(2) action, we can verify in the L^2 norm that

$$\|f\|^2 = \sum_{m=-\infty}^{\infty} \|f_m\|^2, \tag{52}$$

which implies that $L^2(\mathrm{C}^2)$ can be decomposed into the direct sum of L^2_m's;

$$L^2(\mathrm{C}^2) = \bigoplus_{m=-\infty}^{\infty} L^2_m(\mathrm{C}^2), \tag{53}$$

where

$$L^2_m(\mathrm{C}^2) = \{f \in L^2(\mathrm{C}^2) \mid f(e^{is}z) = e^{ims}f(z)\}. \tag{54}$$

We note that $f_m \in L^2_m(\mathrm{C}^2)$.

Our task in the following is to decompose the time evolution (43) in $L^2(\mathrm{C}^2)$ into a series of those in respective subspaces $L^2_m(\mathrm{C}^2)$. This process will be called the reduction of the free particle system for simplicity. We will see later how the time evolution in $L^2_m(\mathrm{C}^2)$ is looked upon as the time evolution of a state on the internal space M. However, before performing the reduction, it is rather necessary to see a few properties of this Green's function and the integral transform. For the purpose of easier reading, we shall write simply $G_t(z,w)$ and $\psi_t(z)$ for $G(z,t;w,t_0)$ and $\psi(z,t)$, respectively. Then $G_t(z,w)$ is invariant under the SO(2) action to the left defined in (32);

$$G_t(e^{is}z, e^{is}w) = G_t(z,w), \tag{55}$$

which is equivalent to the invariance of the Laplacian ∇^2. Further, the integral transform with the Green's kernel $G_t(z,w)$ has the following property for any $s \in \mathrm{R}$

$$\int_{\mathrm{C}^2} G(z,w)\psi_0(w)dw = \int_{\mathrm{C}^2} G(z, e^{-is}w)\psi_0(e^{-is}w)dw, \tag{56}$$

since the Lebesgue measure dw is invaraint under the U(1) action, $w \mapsto e^{is}w$. For $\psi_0 \in L^2(\mathrm{C}^2)$ given, $\psi_0(e^{is}w)$ can be expanded into Fourier series,

$$\psi_0(e^{is}w) = \sum_{m=-\infty}^{\infty} \psi_0^m(w)e^{ims} \quad \text{where} \quad \psi_0^m(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_0(e^{is}w)e^{-ims}ds. \tag{57}$$

Using the above properties and the expansion, we obtain

$$\begin{aligned}
\psi_t(z) &= \int_{\mathbb{C}^2} G_t(z,w)\psi_0(w)dw & (58)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} ds \int_{\mathbb{C}^2} G_t(z,w)\psi_0(w)dw & (59)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} ds \int_{\mathbb{C}^2} G_t(z,e^{-is}w)\psi_0(e^{-is}w)dw & (60)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} ds \int_{\mathbb{C}^2} G_t(z,e^{-is}w)\sum_{m=-\infty}^{\infty} \psi_0^m(w)e^{-ims}dw & (61)\\
&= \sum_{m=-\infty}^{\infty}\int_{\mathbb{C}^2} G_t^m(z,w)\psi_0^m(w)dw, & (62)
\end{aligned}$$

where

$$G_t^m(z,w) := \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{is}z,w)e^{-ims}ds, \qquad (63)$$

and assuming that the order of integration and summation can be interchanged safely, which is the case for $f \in \mathcal{S}(\mathbb{C}^2)$, rapidly decreasing C^∞ functions. The $G_t^m(z,w)$ in (63) is the Green's kernel which operates on $L_m^2(\mathbb{C}^2)$.. At a glance of $G_t^m(z,w)$, we may expect it to have properties that are likes of equivariance. In fact, we can show that

$$\begin{aligned}
G_t^m(e^{is}z,w) &= e^{ims}G_t^m(z,w), & (64)\\
G_t^m(z,e^{is}w) &= e^{-ims}G_t^m(z,w), & (65)
\end{aligned}$$

which means that $G_t^m(z,w)$ is equivariant with respect to the U(1) action on z, and is anti-equivariant with respect to that on w. Thus we have decomposed the time evolution of the original system in $L^2(\mathbb{C}^2)$ into a series of those in $L_m^2(\mathbb{C}^2)$, accomplishing the reduction, as is expressed in (62).

Carrying on from the last passage, (63) can be explicitely computed to give

$$G_t^m(z,w) = \frac{e^{im(\theta(z,w)-\frac{\pi}{2})}}{(2\pi i\hbar(t-t_0))^2}\exp\left(\frac{iB(z,w)}{2\hbar(t-t_0)}\right)J_m\left(\frac{A(z,w)}{2\hbar(t-t_0)}\right), \qquad (66)$$

where J_m is the Bessel function and

$$B(z,w) = \sum_{j=1}^{2}(|z_j|^2+|w_j|^2) \quad \text{for} \quad z_j, w_j \in \mathrm{C}, \tag{67}$$

$$A(z,w) = 2|z_1\overline{w}_1 + z_2\overline{w}_2|, \tag{68}$$

$$\theta(z,w) = \arg\sum_{j=1}^{2} z_j\overline{w}_j. \tag{69}$$

It is of great interst to observe that $A(z,w)$ and $B(z,w)$ can be expressed in the coordinates of the internal space M. In fact, we can verify that

$$B(z,w) = \sqrt{\sum_{k=1}^{3}\xi_k^2} + \sqrt{\sum_{k=1}^{3}\xi_k'^2}, \tag{70}$$

$$A(z,w) = \left[\frac{1}{2}\sqrt{\sum_{k=1}^{3}\xi_k^2}\sqrt{\sum_{k=1}^{3}\xi_k'^2} + \frac{1}{2}\sum_{k=1}^{3}\xi_k\xi_k'\right]^{1/2}, \tag{71}$$

where ξ_k' are given by the formulae

$$\xi_1' + i\xi_2' = 2w_1\overline{w}_2, \tag{72}$$

$$\xi_3' = |w_1|^2 - |w_2|^2. \tag{73}$$

We notice further that under the SO(2) action $z \mapsto e^{is}z$ (resp., $w \mapsto e^{is}w$), the factor $e^{im\theta(z,w)}$ is subject to the transformation $e^{im\theta(z,w)} \mapsto e^{ims}e^{im\theta(z,w)}$ (resp., $e^{im\theta(z,w)} \mapsto e^{-ims}e^{im\theta(z,w)}$).

7 Symmetry due to Particle Exchanges

This section deals with identical particles. According to whether particles are all bosons or fermions, the wave function must be symmetric or antisymmetric with respect to a particle interchange. For our three-particle system, according as paprticles are all bosons or fermions, the wave function ψ on the center-of-mass sytem must satisfy

$$\psi(\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \boldsymbol{x}_{\sigma(3)}) = \psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \quad \text{for} \quad \sigma \in S_3, \quad \text{or} \tag{74}$$

$$\psi(\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \boldsymbol{x}_{\sigma(3)}) = \mathrm{sgn}(\sigma)\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \quad \text{for} \quad \sigma \in S_3, \tag{75}$$

where sgn denotes the signum of σ; $\mathrm{sgn}(\sigma)$ equals 1 of -1, depending on whether σ is an even or odd permutation. In particular, a function satisfying (75) is called satisfying the Pauli principle. For a wave function ψ on the configuration space C^2, one can construct a wave function satisfying the above respective symmetry by the following procedures,

$$\psi^{(s)}(x) := \sum_{h\in S_3} \psi(x\rho(h)^{-1}), \tag{76}$$

$$\psi^{(a)}(x) := \sum_{h\in S_3} \mathrm{sgn}(h)\psi(x\rho(h)^{-1}), \tag{77}$$

where ρ is the representation of S_3 in O(2). The ψ_s and ψ_a indeed satisfy (74) and (75), respectively. We note that $G_t(z,w)$ is invariant under the action of $h \in S_3$ on account of the very definition (44) together with (41);

$$G_t(z\rho(h)^{-1}, w\rho(h)^{-1}) = G_t(z,w), \tag{78}$$

which is equivalent to the invariance of the Laplacian ∇^2 under the action of S_3. This invariance together with the invariance of the Lebesgue measure dw under the action of S_3 will imply that the time evolution preserves the statistics to which the particles are subject, that is, bosonic or fermionic state remains preserved during the time evolution. In fact, for the action of S_3, the time evolution (58) of the initial state undergoes the change

$$\begin{aligned} \psi_t(z\rho(h)^{-1}) &= \int_{\mathrm{C}^2} G_t(z\rho(h)^{-1}, w)\psi_0(w)dw && (79)\\ &= \int_{\mathrm{C}^2} G_t(z, w\rho(h))\psi_0(w)dw && (80)\\ &= \int_{\mathrm{C}^2} G_t(z,w)\psi_0(w\rho(h)^{-1})dw, && (81) \end{aligned}$$

which implies that according to whether $\psi_0(h\rho(h)^{-1}) = \psi_0(z)$ or $\psi_0(z\rho(h)^{-1}) = \mathrm{sgn}(h)\psi_0(z)$ initially, we have for all time t

$$\psi_t(z\rho(h)^{-1}) = \psi_t(z) \quad \text{or} \quad \psi_t(z\rho(h)^{-1}) = \mathrm{sgn}(h)\psi_t(z). \tag{82}$$

Since the action of SO(2) and of S_3 commute, the time evolution (81) is decomposed into

$$\psi_t(z\rho(h)^{-1}) = \sum_{m=-\infty}^{\infty} \psi_t^m(z\rho(h)^{-1}) = \sum_{m=-\infty}^{\infty} \int_{\mathrm{C}^2} G_t^m(z,w)\psi_0^m(w\rho(h)^{-1})dw. \tag{83}$$

Putting (83) together with (76) and (77), we obtain the time evolution of bose or fermi particles in the form,

$$\psi_t^{(s)}(z) = \sum_{m=-\infty}^{\infty} \sum_{h\in S_3} \psi_t^m(z\rho(h)^{-1}), \tag{84}$$

$$\psi_t^{(a)}(z) = \sum_{m=-\infty}^{\infty} \sum_{h\in S_3} \mathrm{sgn}(h)\psi_t^m(z\rho(h)^{-1}) \tag{85}$$

respectively.

8 Complex line bundles

The time evolution ψ_t in $L^2(\mathrm{C}^2)$ was decomposed into the series of those in $L^2_m(\mathrm{C}^2)$,

$$\psi_t^m(z) := \int_{\mathrm{C}^2} G_t^m(z,w)\psi_0^m(w)dw, \quad \psi_0^m \in L^2_m(\mathrm{C}^2). \tag{86}$$

Since G_t^m and ψ_0^m are anti-equivariant and quivariant, respectively, under the SO(2) action $w \mapsto e^{is}w$, the integrand in (86) is invariant under the SO(2) action, so that the integration with respect to w over C^2 will reduce to that over the internal space M. Hence the time evolution $\psi_t^m(z)$ may define the time evolution of a quantum state on the internal space M. To discuss this evolution strictly, we must introduce the notion of complex line bundles.

For a unitary irreducible representation ρ_m of $\mathrm{SO}(2) \cong \mathrm{U}(1)$, $\rho_m(e^{is}) = e^{ims}$, the complex line bundle E_m associated with the SO(2) bundle $\dot{X}_0 \cong \dot{\mathrm{C}}^2 \to M$ is defined to be the quotient of the product space $\dot{X}_0 \times \mathrm{C}$ by the equivalence relation defined through $(z,\zeta) \sim (e^{is}z, e^{ims}\zeta)$ for $(z,\zeta) \in \dot{\mathrm{C}}^2 \times \mathrm{C}$. By $[(z,\zeta)]$ and by π_m we denote the equivalence class in E_m and the projection $E_m \to M$, respectively, so that one has $\pi_m([(z,\zeta)]) = \pi(z)$. A section σ in E_m is a map $M \to E_m$ such that $\pi_m \circ \sigma = \mathrm{id}_M$, where id_M is the identity map of M. Then any ρ_m-equivariant function f on $\dot{X}_0$ determines a section σ in E_m by

$$\sigma(\pi(z)) = [(z, f(z))]. \tag{87}$$

Sections and equivariant functions are in one-to-one correspondence. For sections σ_1, σ_2 corresponding to respective ρ_m-equivariant functions f_1, f_2, the inner product $\langle\sigma_1,\sigma_2\rangle$ is defined through

$$\langle\sigma_1,\sigma_2\rangle = \int_M (\sigma_1,\sigma_2)d\mu_M = \int_{\mathrm{C}^2} \overline{f_1(z)}f_2(z)dz, \tag{88}$$

where (σ_1,σ_2) denotes the inner product in each fiber $\pi_m^{-1}(\pi(z)) \cong \mathrm{C}$, and $d\mu_M$ is the measure on M defined for any function χ on M through the equation

$$\int_M \chi(p)d\mu_M = \int_{\mathrm{C}^2} \chi(\pi(z))dz \quad \text{with} \quad \pi(z) = p. \tag{89}$$

By the definition of the inner product for sections, we see that any function $f \in L_m^2(\mathrm{C}^2)$ determines a square integrable section in E_m. For the equivariant function $\psi_t^m(z)$ given in (86), one has the time evolution of the corresponding section σ_t^m in E_m,

$$\sigma_t^m(\pi(z)) = [(z,\psi_t^m(z))]. \tag{90}$$

Since the time evolution ψ_t is unitary, that is, $\|\psi_t\| = \|\psi_0\|$, in particular, $\|\psi_t^m\| = \|\psi_0^m\|$, the time evolution of the corresponding section σ_t^m is also unitary, that is, $\|\sigma_t^m\| = \|\sigma_0^m\|$ for all time t.

The S_3 action on $L_m^2(\mathrm{C}^2)$ can be transferred to that on square integrable sections in E_m. From (84) and (85), we obtain corresponding time evolutions in E_m, respectively,

$$\sum_{h\in S_3} \sigma_t^m(\pi(z)\tau(h)^{-1}), \tag{91}$$

$$\sum_{h\in S_3} \mathrm{sgn}(h)\sigma_t^m(\pi(z)\tau(h)^{-1}). \tag{92}$$

The reduction is thus completed for the time evolution of free three identical particles on a plane. We have to stress here that we have made full use of the symmetry arising from both the rotation and the particle exchanges in order to obtain the above equations. We found that the key to the reduction by the action of SO(2) was the formation of $f_m(z) \in L_m^2(\mathrm{C}^2)$, which was obtained by operating U(1) on z and then integrating $f(e^{is}z)e^{-ims}$ with respect to the group variable s, as was seen in (49), and that f_m is in-one-one correspondence with a section in E_m which describes a quantum state on the internal space M. Each of Eq. (76) and (77) is a variant of the procedure of the formation of f_m. In fact, particle exchanges were performed by first operating $h \in S_3$ on z and then instead of integrating, discrete sum was taken for $\psi(\sigma(z)\tau(h)^{-1})$ or for $\mathrm{sgn}(h)\psi(\sigma(z)\tau(h)^{-1})$. We may say that this procedure is a form of reduction, while no degrees of freedom are lowered. These two procedures have been put together to yield (91) and (92).

9 Remarks

Let us be reminded that ψ_t^m, which is put in the integral transform given by (86), has determined σ_t^m, which describes the time evolution of a quantum state on the internal space M, as is seen in (90). In view of this, we would like to attempt to put (90) in the following integral transform,

$$\begin{aligned} \sigma_t^m(\pi(z)) &= \left[(z, \int_{\mathrm{C}^2} G_t^m(z,w)\psi_0^m(w)dw) \right] && (93) \\ &= \int_M K_t^m(\pi(z),\pi(w))\sigma_0^m(\pi(w))d\mu_M. && (94) \end{aligned}$$

However, the integral transform in (94) is purely symbolical. In fact, the existence of the Green kernel K_t^m and the way to define integrals for sections are not sure yet. In spite of this, we have already observed in Sec. 8 that the time evolution σ_t^m is unitary, so that we see that there exists a unitary operator U_t^m such that $\sigma_t^m = U_t^m\sigma_0^m$, where U_t^m acts on square integrable sections in E_m.

In conclusion, we try to express the integral transform (86) as an integral on the internal space M explicitly. To this end, we use local sections in the SO(2) bundle $\dot{X}_0 \to M$, which are defined to be

$$\sigma_+(\xi) = \left(\frac{\sqrt{r+\xi_3}}{\sqrt{2}}, \frac{\xi_1 - i\xi_2}{\sqrt{2(r+\xi_3)}} \right) \quad \text{for} \quad \pi(z) = \xi \in D_+, \qquad (95)$$

$$\sigma_-(\xi) = \left(\frac{\xi_1 + i\xi_2}{\sqrt{2(r-\xi_3)}}, \frac{\sqrt{r-\xi_3}}{\sqrt{2}} \right) \quad \text{for} \quad \pi(z) = \xi \in D_-, \qquad (96)$$

where $r^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$, and $D_\pm$ are domains in M defined, respectively, to be

$$D_+ = \{\xi \in \dot{\mathrm{R}}^3 |\, \xi_3 + r \neq 0\}, \qquad (97)$$

$$D_- = \{\xi \in \dot{\mathrm{R}}^3 |\, \xi_3 - r \neq 0\}. \qquad (98)$$

In the intersection $D_+ \cap D_-$, one has the transformation

$$\sigma_-(\xi) = \frac{\xi_1 + i\xi_2}{\sqrt{\xi_1 + \xi_2^2}}\sigma_+(\xi), \quad \xi \in D_+ \cap D_-. \tag{99}$$

By using the section σ_+, points of $\pi^{-1}(D_+)$ are expressed as $z = e^{i\phi}\sigma_+(\xi)$ with $\xi = \pi(z)$, ϕ being an angle variable. Then, for a ρ_m-equivariant function f on $\pi^{-1}(D_+)$, we obtain $f(z) = e^{im\phi}f(\sigma_+(\xi))$, an expression in terms of local coordinates (ξ_j, ϕ) in $\pi^{-1}(D_+)$.

We first devide M into a disjoint union $M = M_+ \cup M_-$, where $M_\pm$ are the upper and the lower half space of $M \cong \dot{\mathrm{R}}^3$. Accordingly, the integral transform (86) is broken up into

$$\begin{aligned} \psi_t^m(z) &= \int_{\pi^{-1}(M_+)} G_t^m(z,w)\psi_0^m(w)dw \ + \int_{\pi^{-1}(M_-)} * \qquad (100)\\ &= \int_{\pi^{-1}(M_+)} F_t^m(\xi,\xi')e^{im\arg\langle z,w\rangle}\psi_0^m(w)dw \ + \int_{\pi^{-1}(M_-)} * \quad , \qquad (101)\end{aligned}$$

where $\langle z, w\rangle = \sum_j z_j\overline{w}_j$ and

$$F_t^m(\xi,\xi') = \frac{e^{-im\pi/2}}{(2\pi i\hbar(t-t_0))^2}\exp\Big(\frac{\widetilde{B}(\xi,\xi')}{2\hbar(t-t_0)}\Big)J_m\Big(\frac{\widetilde{A}(\xi,\xi')}{2\hbar(t-t_0)}\Big), \tag{102}$$

and also $\widetilde{A}(\xi,\xi') = A(z,w)$, $\widetilde{B}(\xi,\xi') = B(z,w)$ on account of (70) and (71). Then we use the local sections σ_+ and σ_- on M_+ and M_-, respectively, to rewrite the last integrals. In particular, for $z \in \pi^{-1}(D_+)$, Eq.(101) results in

$$\begin{aligned} \psi_t^m(\sigma_+(\xi)) &= \int_{M_+} F_t^m(\xi,\xi')e_{++}^m(\xi,\xi')\psi_0^m(\sigma_+(\xi'))d\mu_M(\xi') \\ &\quad + \int_{M_-} F_t^m(\xi,\xi')e_{+-}^m(\xi,\xi')\psi_0^m(\sigma_-(\xi'))d\mu_M(\xi'), \qquad (103)\end{aligned}$$

where

$$\begin{aligned} e_{++}^m(\xi,\xi') &= e^{im\arg\langle\sigma_+(\xi),\sigma_+(\xi')\rangle}, \qquad (104)\\ e_{+-}^m(\xi,\xi') &= e^{im\arg\langle\sigma_+(\xi),\sigma_-(\xi')\rangle}. \qquad (105)\end{aligned}$$

A similar expression for $\psi_t^m(\sigma_-(\xi))$, $\xi \in D_-$ can be obtained as well with due definition of $e_{-+}^m(\xi,\xi')$ and of $e_{--}^m(\xi,\xi')$. The functions $\psi_t^m(\sigma_-(\xi))$ and

$\psi_t^m(\sigma_+(\xi))$ are related on $D_+ \cap D_-$ by

$$\psi_t^m(\sigma_-(\xi)) = \Big(\frac{\xi_1 + i\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}}\Big)^m \psi_t^m(\sigma_+(\xi)), \quad \xi \in D_+ \cup D_-, \tag{106}$$

which is observed from (99) and the fact that ψ_t^m is ρ_m-equivariant. We conclude this section with saying that the section $\sigma_t^m(\xi)$ is expressed as $\sigma_t^m(\xi) = [(\sigma_+(\xi), \psi_t^m(\sigma_+(\xi))]$ for $\xi \in D_+$ and $\sigma_t^m(\xi) = [(\sigma_-(\xi), \psi_t^m(\sigma_-(\xi))]$ for $\xi \in D_-$, respectively.

References

[1] Toshihiro Iwai: *A gauge theory for the quantum planar three-body problem*, J. Math. Phys. 28 (4), April 1987